An introduction to Hybrid High Order (HHO) methods with applications to incompressible fluid flows

Daniel Castanon Quiroz[†] and Daniele A. Di Pietro[‡]

[†]IIMAS-UNAM (National Autonomous University of Mexico), Mexico City, Mexico.
[‡] IMAG (Institut Montpelliérain Alexander Grothendieck), Montpellier, France.

NEMESIS - Kick off Meeting 2024
Montpellier, France

Outline

Outline

1 **Intro to the Hybrid High Order (HHO) method**

[Di Pietro, Ern, and Lemaire 2014] → First introduced
[Di Pietro and Droniou 2020] →An HHO Book with different Apps
[Cicuttin, Ern and Pignet 2021]→ An HHO Book with App. in Solid Mechanics

2 **HHO methods for incompressible fluid flows**

[CQ and Di Pietro 2020] → Pressure-robust Navier-Stokes formulation on simplicial meshes
[CQ and Di Pietro 2023] → Pressure-robust stationary Navier-Stokes formulation on polytopal meshes
[CQ and Di Pietro 2024] → Pressure-robust time-dependent Navier-Stokes formulation on polytopal meshes

Setting: Poisson

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open connected polytopal Lipschitz domain
- We focus on the Poisson problem: Given $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Let $f \in L^2(\Omega)$. The usual weak formulation reads: Find $u \in H_0^1(\Omega)$ s.t.

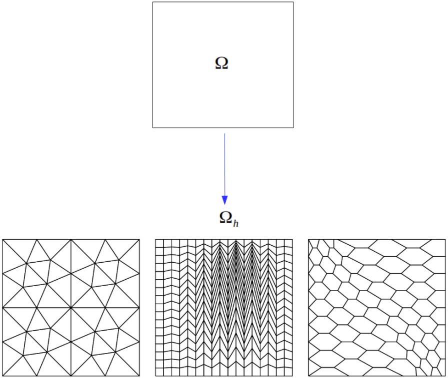
$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega)$$

- The well-posedness of this problem hinges on the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega)^d} \quad \forall v \in H_0^1(\Omega)$$

Poisson Problem.

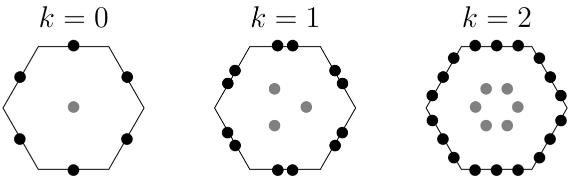
- Discretisation of Ω to Ω_h



HHO in a Nutshell

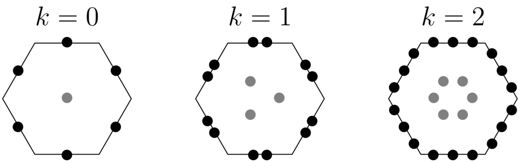
Hybrid High Order (HHO) in a nutshell

- HHO methods attach discrete unknowns to **mesh faces**
  - one polynomial of order $k \geq 0$ on each mesh face
- HHO methods also use **element unknowns**
  - one polynomial of order $k \geq 0$ on each mesh element
  - elimination by static condensation



Ex: Degrees of Freedom (DOFs) using HHO with hexagonal elements for the scalar case

HHO Dofs



- Let $k \geq 0$ and define the local Hybrid High-Order (HHO) space

$$\underline{V}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \; : \; v_T \in \mathcal{P}^k(T) \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_T \right\}$$

- Given a polytopal mesh $\mathcal{T}_h$ of Ω, define the global HHO space

$$\underline{V}_h^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) : \\ v_T \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \right\}$$

Discrete Poincaré inequality in HHO spaces I

- We define on $\underline{V}_h^k$ the H^1-like seminorm

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$$

$$\text{where } \|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F\in\mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \quad \text{for all } T \in \mathcal{T}_h$$

Lemma (Discrete Poincaré inequality in HHO spaces)

Denote by $\underline{V}_{h,0}^k$ the subspace of $\underline{V}_h^k$ with vanishing boundary values. For any $\underline{v}_h \in \underline{V}_{h,0}^k$, letting $v_h \in \mathcal{P}^k(\mathcal{T}_h)$ be s.t. $(v_h)_{|T} := v_T$ for all $T \in \mathcal{T}_h$,

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h},$$

hence $\|\cdot\|_{1,h}$ is a norm on $\underline{V}_{h,0}^k$.

Discrete local operators

- For $k \geq 0$ we define the reconstruction operator $r_T^{k+1} : \underline{V}_T^k \to \mathcal{P}^{k+1}(T)$ s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\int_T \nabla r_T^{k+1} \underline{v}_T \cdot \nabla w = -\int_T v_T \, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\int_T r_T^{k+1} \underline{v}_T = \int_T v_T$$

We consider the following scheme:

- Find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

where, for all $T \in \mathcal{T}_h$,

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla r_T^{k+1} \underline{u}_T \cdot \nabla r_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

and the symmetric semi-definite stabilization bilinear form s_T satisfies

$$\boxed{\|\underline{v}_T\|_{1,T}^2 \lesssim a_T(\underline{v}_T, \underline{v}_T) \lesssim \|\underline{v}_T\|_{1,T}^2 \quad \forall \underline{v}_T \in \underline{V}_T^k} \qquad \text{(ST1)}$$

Error analysis I

- Let $\underline{I}_h^k : H^1(\Omega) \to \underline{V}_h^k$ be the HHO interpolator s.t.

$$\underline{I}_h^k v := ((\pi_{\mathcal{P}^k(T)} v)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P}^k(F)} v)_{F \in \mathcal{F}_h}) \quad \forall v \in H^1(\Omega)$$

- For estimating the error

$$\underline{e}_h := \underline{u}_h - \underline{I}_h^k u \in \underline{V}_{h,0}^k$$

- We define the consistency error $\mathcal{E}_h(u; \underline{v}_h)$ for all $\underline{v}_h \in \underline{V}_{h,0}^k$ such that

$$\mathcal{E}_h(u; \underline{v}_h) := \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h)$$

- We use the 3rd Strang Lemma (J. Droniou's presentation):

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \leq \sup_{\underline{v}_h \in \underline{V}_{h,0}^k \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(u; \underline{v}_h)}{\|\underline{v}_h\|_{1,h}}$$

Error analysis II

Doing some algebra, we get

$$\mathcal{E}_h(u;\underline{v}_h) = \cancel{\sum_{T\in\mathcal{T}_h} \int_T \left[\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)\right]\cdot\nabla v_T} \quad \underbrace{+ \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F \left[\nabla u - (\nabla r_T^{k+1}(\underline{I}_T^k u)\cdot n_{TF})\right](v_F - v_T)}_{\mathfrak{T}_1}$$

$$\underbrace{-\sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)}_{\mathfrak{T}_2}$$

Using Cauchy–Schwarz inequalities, definition of $\|\cdot\|_{1,h}$, and optimal approximation properties of $\pi_{\mathcal{P}^{k+1}(T)}$ we get

$$\mathfrak{T}_1 \lesssim h^{k+1}|u|_{H^{k+2}(\mathcal{T}_h)}\|\underline{v}_h\|_{1,h}$$

To have $\mathfrak{T}_2$ scale as $\mathfrak{T}_1$, we further assume polynomial consistency:

$$\boxed{s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T)\times \underline{V}_T^k} \tag{ST2}$$

Error analysis III

And using again the optimal approximation properties of $\pi_{\mathcal{P}^{k+1}(T)}$, we get

$$\mathfrak{T}_2 \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Theorem (Error estimate for the HHO scheme)

Denote by $u \in H_0^1(\Omega)$ the solution to the Poisson problem and by $\underline{u}_h \in \underline{V}_h^k$ its HHO approximation. Then, under (ST1)–(ST2), *and further assuming $u \in H^{k+2}(\mathcal{T}_h)$, it holds*

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)}.$$

$$\boxed{\|\underline{v}_T\|_{1,T}^2 \lesssim a_T(\underline{v}_T, \underline{v}_T) \lesssim \|\underline{v}_T\|_{1,T}^2 \quad \forall \underline{v}_T \in \underline{V}_T^k} \tag{ST1}$$

$$\boxed{s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k} \tag{ST2}$$

The Stokes Problem

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded, simply connected polytopal domain with Lipschitz boundary $\partial\Omega$
- Given a body force $\boldsymbol{f} \in L^2(\Omega)^d$. We consider the problem:

  Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)^d$ such that

$$\begin{aligned} \nu a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \ell(\boldsymbol{f}, \boldsymbol{v}) \quad && \forall \boldsymbol{v} \in H_0^1(\Omega)^d, \\ -b(\boldsymbol{u}, q) &= 0 && \forall q \in L^2(\Omega)^d, \end{aligned}$$

  where $\nu > 0$ is the fluid viscosity, and

$$a(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v}, \quad b(\boldsymbol{v}, q) := -\int_\Omega (\nabla\cdot \boldsymbol{v}) q, \quad \ell(\boldsymbol{f}, \boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}$$

Motivation: Robust Numerical Methods

- We call a numerical method "pressure-robust" ([Linke 2014]) if the discretisation error of the velocity is "independent of the pressure", i.e.,

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^2(\Omega)^d} \leq Ch^r \|\boldsymbol{u}\|_{H^s(\Omega)^d},$$

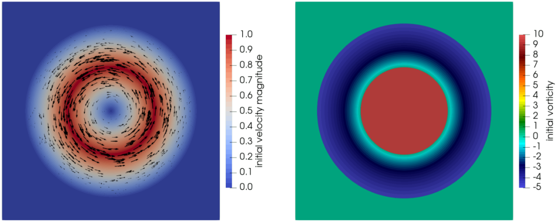
  where $\boldsymbol{u}_h$ is the approximation of the solution $\boldsymbol{u}$, h is the mesh size, C is a constant independent of the pressure p, and r, s are positive integers

- If $\boldsymbol{f} = \nabla\phi$ for $\phi \in H^1(\Omega)$, then this body force is absorbed by the pressure gradient, not by $\boldsymbol{u}$

Motivation: Pressure Robustness

Example: The Gresho vortex problem with translation in $\mathbb{R}^2$

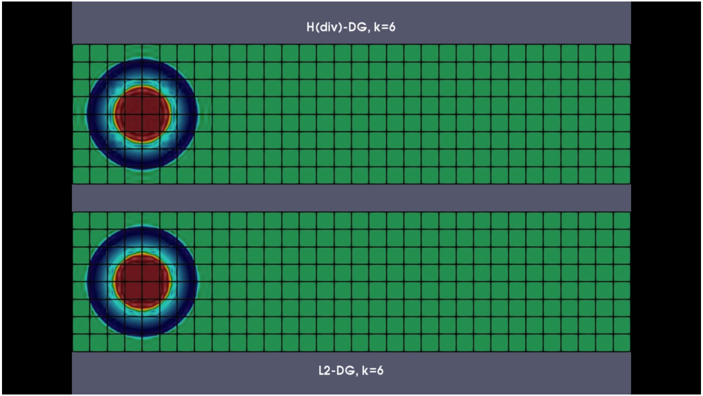
- Time dependent problem.
- The initial condition is constructed such that:
  - The vorticity $\nabla \times \boldsymbol{u}$ is always constant
  - The field force is $\boldsymbol{f} = \boldsymbol{0}$, and $\nu = 10^{-5}$
- Periodic boundary conditions
- Example/simulation taken from [Gauger *et. al.* 2019] and [Schroeder 2019]



Initial condition: $\boldsymbol{u}_0$ and its vorticity $\nabla \times \boldsymbol{u}_0$

Motivation: Pressure Robustness

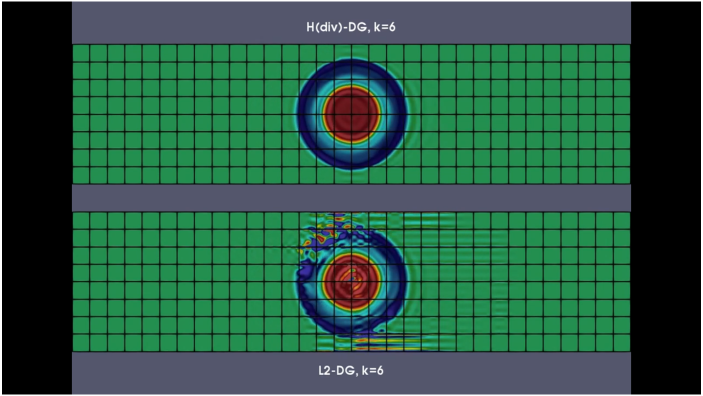
Example: The Gresho vortex problem with translation in $\mathbb{R}^2$



Vorticity $\nabla \times \boldsymbol{u}$ at $t = 0$

Motivation: Pressure Robustness

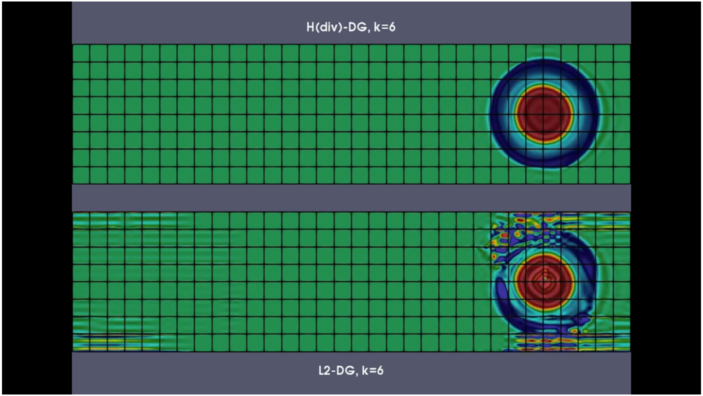
Example: The Gresho vortex problem with translation in $\mathbb{R}^2$



Vorticity $\nabla \times \boldsymbol{u}$ at $t = 1.5$

Motivation: Pressure Robustness

Example: The Gresho vortex problem with translation in $\mathbb{R}^2$



Vorticity $\nabla \times \boldsymbol{u}$ at $t = 3$

The HHO Space

Objective

To design HHO discretization methods on **general meshes** for incompressible fluid problems

such that the velocity error estimates are **independent** from the pressure

The HHO Space

The HHO Space

- The global spaces of discrete unknowns for the velocity and the pressure, respectively accounting for the wall boundary condition and the zero-average condition, are

$$\underline{\boldsymbol{U}}_{h,0}^{k} := \left\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}) \in \underline{\boldsymbol{U}}_h^k : \boldsymbol{v}_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\},$$
$$P_{h,0}^{k} := \mathcal{P}^k(\mathcal{T}_h) \cap L_0^2(\Omega)^d$$

Local Pressure-Velocity Coupling I

- Let an element $T \in \mathcal{T}_h$ be fixed. We define the local discrete divergence operator $D_T^k : \underline{\boldsymbol{U}}_T^k \to \mathcal{P}^k(T)$ as follows:

  For $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$, $D_T^k \underline{\boldsymbol{v}}_T$ is such that, for all $q \in \mathcal{P}^k(T)$,

$$\int_T D_T^k \underline{\boldsymbol{v}}_T q = -\int_T \boldsymbol{v}_T \cdot \nabla q + \sum_{F \in \mathcal{F}_T} \int_F \boldsymbol{v}_F \cdot \boldsymbol{n}_{TF} q$$

- The operator D_T^k satisfies the commutating property

$$D_T^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v} = \pi_T^k (\nabla \cdot \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in H^1(T)^d$$

Local Pressure-Velocity Coupling II

- For the pressure-velocity coupling, we define the bilinear form $b_h : \underline{\boldsymbol{U}}_h^k \times P_{h,0}^k(\mathcal{T}_h) \to \mathbb{R}$ such that

$$b_h(\underline{\boldsymbol{v}}_h, q_h) := \sum_{T \in \mathcal{T}_h} \int_T -(D_T^k \underline{\boldsymbol{v}}_h)\, q_h$$

- *Stability*. It holds, for all $q \in P_{h,0}^k(\mathcal{T}_h)$,

$$\|q\|_{L^2(\Omega)} \lesssim \sup_{\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_{h,0}^k, \|\underline{\boldsymbol{v}}_h\|_{1,h}=1} b_h(\underline{\boldsymbol{v}}_h, q_h)$$

Robustness with respect to pressure

- The weak Stokes problem: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)^d$ such that

$$\nu a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \ell(\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^d,$$
$$-b(\boldsymbol{u}, q) = 0 \quad \forall q \in L^2(\Omega)^d$$

- To make it robust we approximate $\ell(\boldsymbol{f}, \boldsymbol{v})$ by $\ell_h : \boldsymbol{L}^2(\Omega) \times \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ the bilinear form is

$$\ell_h(\boldsymbol{f}, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{f} \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T,$$

  where $\boldsymbol{R}_T^k : \underline{\boldsymbol{U}}_T^k \to$ a conformal subspace of $\mathbf{H}_{\text{div}}(T)$.
- See [Di Pietro, Ern, Linke, and Schieweck 2016] $\to$ HHO robust method for the Stokes problem using simplicial meshes
- Need to extend the above method on polytopal meshes

Velocity Reconstruction in $\mathbf{H}_{\text{div}}(T)$ I

- Let an element $T \in \mathcal{T}_h$ be fixed[1], and let $\mathfrak{T}_T$ a regular simplicial subdivision of T
  For $\tau \in \mathfrak{T}_T$, let $\mathbb{RTN}^k(\tau)$ the local Raviart–Thomas–Nédélec space of degree k
- Restrictions on $\mathfrak{T}_T$:
  - For $k \geq 2$: All simplices in $\mathfrak{T}_T$ have at least one common vertex denoted as $\boldsymbol{x}_T$

  Two examples of submeshes $\mathfrak{T}_T$ in $\mathbb{R}^2$ that satisfy the assumptions above:

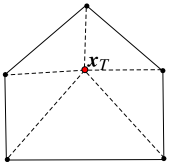


Figure: Pyramidal sub

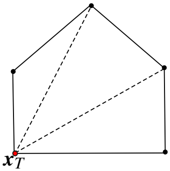


Figure: Non-pyramidal sub

- We denote as τ a simplicial element which belongs to $\mathfrak{T}_T$, and as σ a face of τ
- The simplicial subdivision $\mathfrak{T}_T$, is used to construct local operators for each mesh element T, and will not modify the final size of the global system

[1] We assume T is star-shaped with respect to a ball.

Velocity Reconstruction in $\mathbf{H}_{\text{div}}(T)$ II

- We introduce the following space generated by the Koszul operator[2]:

$$\mathcal{G}^{\mathrm{c},k}(T) := (\boldsymbol{x} - \boldsymbol{x}_T) \times \mathcal{P}^{k-1}(T)^3 \qquad \text{for } k \geq 1,$$

  and define $\mathcal{G}^{\mathrm{c},-1}(T) := \mathcal{G}^{\mathrm{c},0}(T) := \{0\}$

- Observe that we have the decomposition:

$$\mathcal{P}^k(T)^d = \nabla\mathcal{P}^{k+1}(T) \oplus \mathcal{G}^{\mathrm{c},k}(T),$$

  where the direct sum above is not orthogonal in general

[2] See [Di Pietro and Droniou 2021].

Velocity Reconstruction in $\mathbf{H}_{\text{div}}(T)$ III

- We define $\boldsymbol{R}_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{RTN}^k(\mathfrak{T}_T)$ as the first component of the solution of the following local problem:
  Given $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$, find $(\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T, \psi, \boldsymbol{\theta}) \in \mathbb{RTN}^k(\mathfrak{T}_T) \times \mathcal{P}^k(\mathfrak{T}_T) \times \boldsymbol{\mathcal{G}}^{\mathrm{c},k-1}(T)$ such that

$$
\begin{aligned}
\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_\sigma &= (\boldsymbol{v}_F \cdot \boldsymbol{n}_{TF})_{|\sigma} && \forall \sigma \in \mathfrak{F}_F, \forall F \in \mathcal{F}_T, \\
\int_T (\nabla\cdot \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T)\, \phi &= \int_T (D_T^k \underline{\boldsymbol{v}}_T)\, \phi && \forall \phi \in \mathcal{P}^k(\mathfrak{T}_T), \\
\int_T \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{\xi} &= \int_T \boldsymbol{v}_T \cdot \boldsymbol{\xi} && \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{G}}^{\mathrm{c},k-1}(T), \\
\int_T \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \mathfrak{w} + \int_T (\nabla\cdot \mathfrak{w})\, \psi + \int_T \mathfrak{w} \cdot \boldsymbol{\theta} &= \int_T \boldsymbol{v}_T \cdot \mathfrak{w} && \forall \mathfrak{w} \in \mathbb{RTN}_0^k(\mathfrak{T}_T)
\end{aligned}
$$

  where $\mathcal{F}_T$ are the faces of T, $\mathfrak{F}_F$ the subdivision of F, and $\mathbb{RTN}_0^k(\mathfrak{T}_T)$ is the subspace of $\mathbb{RTN}^k(\mathfrak{T}_T)$ with vanishes $\forall F \in \mathcal{F}_T$

(Similar construction in [Frerichs and Merdon 2022] within the conformal VEM framework)

Velocity Reconstruction in $\mathbf{H}_{\text{div}}(T)$ IV

Lemma (Properties of $\boldsymbol{R}_T^k$) [CQ and Di Pietro 2023]

The operator $\boldsymbol{R}_T^k$ has the following properties:

- *Well-posedness and boudedness*. For a given $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$, there exists a unique element $\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \in \mathbb{RTN}^k(\mathfrak{T}_T)$ and it holds that

$$\|\boldsymbol{v}_T - \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T\|_{L^2(T)^3} \lesssim h_T |\underline{\boldsymbol{v}}_T|_{1,\partial T}$$

- *Approximation*. For all $\boldsymbol{v} \in H^{k+1}(T)^d$, we have the bound

$$\|\boldsymbol{v} - \boldsymbol{R}_T^k \underline{\boldsymbol{I}}_T^k \boldsymbol{v}\|_{L^2(T)^3} \lesssim h_T^{k+1} |\boldsymbol{v}|_{H^{k+1}(T)^d}$$

- *Consistency*. For a given $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$, it holds, for $k \geq 1$,

$$\boldsymbol{\pi}_T^{k-1}(\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T) = \boldsymbol{\pi}_T^{k-1}(\boldsymbol{v}_T)$$

The HHO Scheme and the Error Estimate

The Discrete Problem

- The HHO discretization of the Stokes problem then reads:
  Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ such that

$$\nu a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h) = \ell_h(\boldsymbol{f}, \underline{\boldsymbol{v}}_h) \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}, \tag{2.5a}$$

$$-b_h(\underline{\boldsymbol{u}}_h, q_h) = 0 \quad \forall q_h \in \mathcal{P}^k(\mathcal{T}_h). \tag{2.5b}$$

Theorem (Convergence) [CQ and Di Pietro 2023]

Let $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ be a solution to the Stokes equations, and $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_h^k \times P_h^k$ be a solution to the HHO scheme (2.5). Then, it holds:

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_k^h \boldsymbol{u}\|_{1,h} \le C h^{k+1} |\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^d}$$

Extension to Navier–Stokes I

The Discrete Problem

- Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ such that

$$\nu a_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + t_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + b_h(\underline{\boldsymbol{v}}_h, p_h) = \ell_h(\boldsymbol{f}, \underline{\boldsymbol{v}}_h) \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}, \tag{2.6a}$$

$$-b_h(\underline{\boldsymbol{u}}_h, q_h) = 0 \quad \forall q_h \in \mathcal{P}^k(\mathcal{T}_h) \tag{2.6b}$$

where

$$\begin{aligned} t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{z}}_h) = & \sum_{T\in\mathcal{T}_h} \left[\int_T \nabla \boldsymbol{w}_T \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T - \int_T \nabla \boldsymbol{w}_T \boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \right] \\ & + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{w}_F - \boldsymbol{w}_T) \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T \left(\boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \cdot \boldsymbol{n}_{TF} \right) \\ & - \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{w}_F - \boldsymbol{w}_T) \cdot \boldsymbol{R}_T^k \underline{\boldsymbol{v}}_T \left(\boldsymbol{R}_T^k \underline{\boldsymbol{z}}_T \cdot \boldsymbol{n}_{TF} \right) \end{aligned}$$

- We use the key identity $\int_\Omega (\nabla \times \boldsymbol{u}) \times \boldsymbol{v} \cdot \boldsymbol{w} = \int_\Omega ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{w}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v})$
- The form t_h is non-dissipative, i.e., $t_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{v}}_h) = 0$

Extension to Navier–Stokes II

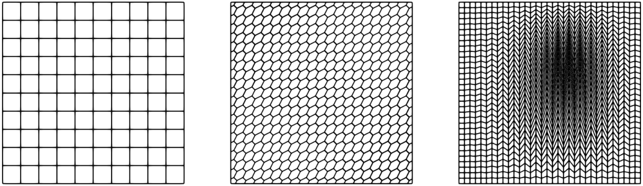
Theorem (Convergence) [CQ and Di Pietro 2023]

Let $(\boldsymbol{u}, p) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ be a solution to the Navier–Stokes equations, and $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}^k_h \times P^k_h$ be a solution to the HHO scheme (2.6). Then, it holds:

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^h_k \boldsymbol{u}\|_{1,h} \le C h^{k+1} \left(|\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h)^d} + \nu^{-1} \|\boldsymbol{u}\|_{W^{1,4}(\Omega)^d} |\boldsymbol{u}|_{W^{k+1,4}(\mathcal{T}_h)^d} \right)$$

Numerical Test: Lid-Driven Cavity I

- Domain: $[0,1] \times [0,1]$. BCs: $\boldsymbol{u} = 0$ at the walls, and $\boldsymbol{u} = (1,0)$ at the top
- Body force $\boldsymbol{f} = 0$
- Using Polynomial aproximation: $k = 1$. Reynolds number $Re = \nu^{-1}$
- $Re = 1,000$
- Using 3 different meshes:



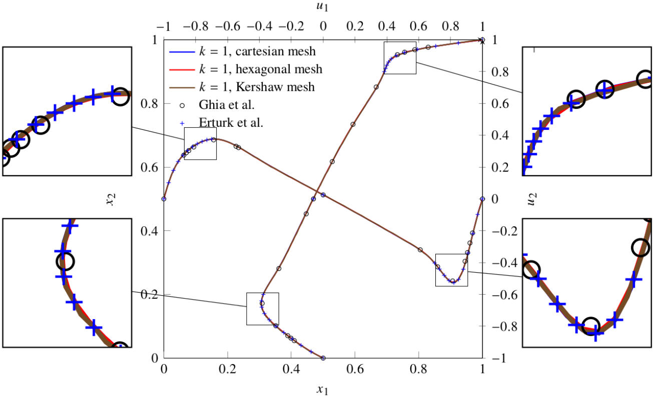
(a) Cartesian. (b) Hexagonal. (c) Kershaw.

Meshes (coarser version) used for the Lid-Driven Cavity

- Number of DOFs after static condensation: Cartesian-58240, Hexa-339521, Kershaw-246345

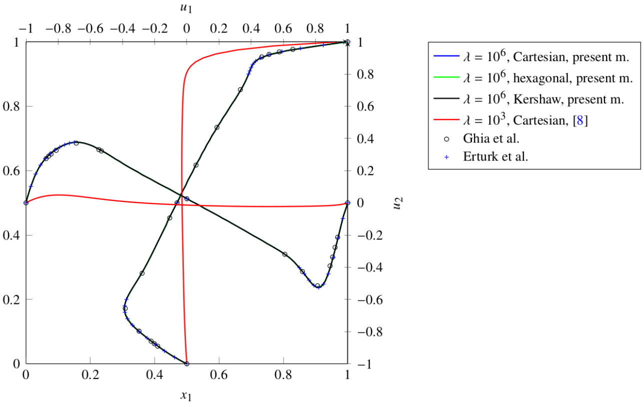
Numerical Test: Lid-Driven Cavity II

- Comparison with reference solution [Erturk, Corke, Gökçöl 2005] and [Ghia, Ghia, Shin 1982] for $Re = 1,000$



Numerical Test: Lid-Driven Cavity III

- To check the robustness of the method we run the same test case for $Re = 1,000$, but with $f = \lambda \nabla \psi$, where $\psi = \frac{1}{3}(x^3 + y^3)$
- Comparison against the original HHO formulation of [Botti, Di Pietro, and Droniou 2019]



Conclusion

- Pressure robustness for the time dependent Navier–Stokes problem: [CQ and Di Pietro 2024] to appear soon!
- Getting pressure robustness without $\mathfrak{I}_T$: Use a variational formulation where $\boldsymbol{u} \in \mathbf{Hcurl}(\Omega)$
    - See [Beirão da Veiga, Dassi, Di Pietro and Droniou 2022] for the Stokes problem using the DDR/VEM frameworks
    - See [Di Pietro, Droniou and Qian 2024] for the Navier–Stokes problem using the DDR framework

Thank you

Thank you for your attention!

You can download this presentation @ danielcq-math.github.io

References I

[Botti, Di Pietro, Droniou 2019] Botti, L., Di Pietro, D. A., and Droniou, J.
A Hybrid High-Order method for the incompressible Navier-Stokes equations based on Temam's device.
Journal of Computational Physics, 2019.

[Beirão da Veiga *et. al.* 2022] L. Beirão da Veiga, F. Dassi, D. A. Di Pietro, and J. Droniou
Arbitrary-order pressure-robust DDR and VEM methods for the Stokes problem on polyhedral meshes .
Comput. Meth. Appl. Mech. Engrg., 2022, 397(115061).

[CQ and Di Pietro 2020] Castanon Quiroz, D., and Di Pietro, D. A.
A Hybrid High-Order method for the incompressible Navier-Stokes problem robust for large irrotational body forces.
Comput. Math. Appl.,79-9, 2020.
https://doi.org/10.1016/j.camwa.2019.12.005

[CQ and Di Pietro 2023] Castanon Quiroz, D., and Di Pietro, D. A.
A pressure-robust HHO method for the solution of the incompressible Navier-Stokes equations on general meshes.
IMA Journal of Numerical Analysis, published online April 2023.
https://doi.org/10.1093/imanum/drad007

[CQ and Di Pietro 2024] Castanon Quiroz, D., and Di Pietro, D. A.
A Reynolds-semi-robust and pressure robust Hybrid High-Order method for the time dependent incompressible Navier–Stokes equations on general meshes.
In Preparation.

[Cicuttin, Ern and Pignet 2021] Cicuttin, M., Ern, A., and Pignet, N.
Hybrid high-order methods. A primer with application to solid mechanics.
SpingerBriefs in Mathematics, 2021.

References II

[Di Pietro and Droniou 2020] Di Pietro, D. A., and Droniou, J.
The Hybrid High-Order Method for Polytopal Meshes - Design, Analysis and Applications.
Number 19 in Modeling, Simulation and Applications Springer International Publishing, 2020.

[Di Pietro and Droniou 2021] Di Pietro, D. A., and Droniou, J.
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincare inequalities, and consistency .
Found. Comput. Math. (2021).

[Di Pietro, Droniou and Qian 2024] D. A. Di Pietro, J. Droniou, and J. J. Qian
A pressure-robust Discrete de Rham scheme for the Navier–Stokes equations .
Comput. Meth. Appl. Mech. Engrg., 2024, 421(116765).

[Di Pietro, Ern, Lemaire 2014] Di Pietro, D. A., Ern, A., and Lemaire, S.
An Arbitrary-Order and Compact-Stencil Discretization of Diffusion on General Meshes Based on Local Reconstruction Operators.
Computational Methods in Applied Mathematics, 2014.

[Di Pietro, Ern, Linke, and Schieweck 2016] Di Pietro, D. A., Ern, A., Linke, A., and Schieweck, F.
A discontinuous skeletal method for the viscosity-dependent Stokes problem.
Computer Methods in Applied Mechanics and Engineering, vol. 306, 2016.

[Erturk, Corke, Gökçöl 2005] Erturk, E., Corke, T. C., and Gökçöl, C.
Numerical solutions of 2-D steady incompressible driven cavity flow at high Reynolds.
Int. J. Numer. Meth. Fluids, 2005.

[Frerichs and Merdon 2022] Frerichs, D., and Merdon, C.
Divergence-preserving reconstructions on polygons and a really pressure-robust virtual element method for the Stokes problem.
IMA Journal of Numerical Analysis, Volume 42, Issue 1, January 2022.

References III

[Gauger *et. al. 2019] Gauger, N.R., Linke, A. and Schroeder, P.W.*
On high-order pressure-robust space discretisations, their advantages for incompressible high Reynolds number generalised Beltrami flows and beyond .
SMAI Journal of Computational Mathematics. Volume 5 (2019), p. 89-129.

[Ghia, Ghia, Shin 1982] Ghia, U., Ghia, K.N., and Shin, C.T.
High-Re solutions for incompressible flow using the Navier–Stokes equations and a multigrid method.
J. Comput. Phys., 1982.

[Linke 2014] Linke, A.
On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime.
Comput. Methods Appl. Mech. Engrg. 268 (2014) 782–800.

[Schroeder 2019] Schroeder, P.W.
Robustness of High-Order Divergence-Free Finite Element Methods for Incompressible Computational Fluid Dynamics
PhD Dissertation, Georg-August-Universität Göttingen, 2019.